SOLUTION OF A DYNAMIC PROBLEM WITH MIXED BOUNDARY CONDITIONS IN THE THEORY OF ELASTICITY FOR A HALF-PLANE

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PMM Vol.23, No.2, 1959, pp. 266-273

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(Received 15 April 1957)

This article considers a problem describing the dynamic response of an elastic half-plane to the impact of a system of stamps in the absence of frinction and cohesion.

It is assumed that the reader is familiar with Sobolev's results, contained in Sections 3-4, chapter 12, [1].

1. Uniqueness theorem. The following basic mixed boundary value problem will be considered. We are given a set, $L = L_1 + \ldots + L_n$, of intervals (a_k, b_k) arranged along the positive half of the x-axis in such a way that the endpoints of these intervals form a sequence $(a_1, b_1, a_2, b_2, \ldots, a_n, b_n)$. The following displacements are given on these intervals:

$$u(x, 0, t) = f_1(x, t) + d_1(x, t), \qquad v(x, 0, t) = f_2(x, t) + d_2(x, t) \quad (1.1)$$

where $d_i(x, t) = d_i(t)$ $(i = 1, 2)$ on L. We are also given the principal vector (X^0, Y^0) of external forces applied to L (problem A), or $d_i(x, t) = d_{ik}(t)$ $(k = 1, \ldots, n)$ on the segments L_k, and the principal vectors (X_k^0, Y_k^0) of external forces applied to each of the segments L_k (problem B). On the remaining part L' of the boundary are given the stresses

$$\sigma_y(x, 0, t) = A(x, t), \qquad \tau_{xy}(x, 0, t) = B(x, t) \quad (1.2)$$

In addition, we are given the body forces X, Y and the initial conditions.

$$\begin{aligned} u(x, y, 0) &= u_0(x, y), \qquad \left(\frac{\partial u}{\partial t}\right)_{t=0} = u_0'(x, y) \\ v(x, y, 0) &= v_0(x, y), \qquad \left(\frac{\partial v}{\partial t}\right)_{t=0} = v_0'(x, y) \end{aligned} \quad (1.3)$$

Problem A and B as formulated can easily be reduced to a system of integral equations for stresses σ_y and τ_{xy} on the segment L of the x-axis. For this purpose, on the basis of a result given in reference [1], we may write the following equations

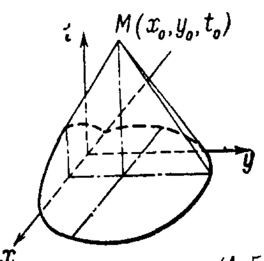
$$2\pi\int_0^{t_0}(t_0-t)\left(\sigma_{y_0}+2\mu\frac{\partial u}{\partial x_0}\right)dt = M(x_0, y_0, t_0)$$
$$2\pi\int_0^{t_0}(t_0-t)\left(\tau_{x,y_0}-2\mu\frac{\partial v}{\partial x_0}\right)dt = N(x_0, y_0, t_0) \qquad (1.4)$$

where

$$M(x_0, y_0, t_0) = -\iiint_T(u_1{}^\circ X + v_1{}^\circ Y)\,d\tau + \iint_S(u_1{}^\circ\tau_{xy}+v_1{}^\circ\sigma_y)\,dxdt +$$
$$+\rho\iint_{S_1}\left(u\frac{\partial u_1{}^\circ}{\partial t}+v\frac{\partial V_1{}^\circ}{\partial t}-u_1{}^\circ\frac{\partial u}{\partial t}-v_1{}^\circ\frac{\partial v}{\partial t}\right)dxdy$$

We obtain a similar equation for N from the expression of M if we replace the fundamental solution $u_1{}^0$, $v_1{}^0$ of the longitudinal type by the fundamental solution $u_2{}^0$, $v_2{}^0$ of the transverse type. The volume T is bounded by the surface of the characteristic cone and by the planes $y = 0$, $t = 0$, as shown in the figure. In equation (1.4) we let y_0 go to zero, and introducing the notation $P(X, t) = \sigma_y(X, 0, t)$, $Q(x, t) = \tau_{xy}(x, 0, t)$, we obtain

$$2\pi\int_0^{t_0}(t_0-t)P(x_0, t)\,dt = \iint_S[u_1{}^\circ|_{y_0=0}Q(x, t)+$$
$$+v_1{}^\circ|_{y_0=0}P(x, t)]\,dxdt+\Phi_1(x_0, t_0)$$
$$2\pi\int_0^{t_0}(t_0-t)Q(x_0, t)\,dt = \iint_S[u_2{}^\circ|_{y_0=0}Q(x, t)+$$
$$+v_2{}^\circ|_{y_0=0}P(x, t)]\,dxdt+\Phi_2(x_0, t_0) \qquad (1.5)$$



Here the expressions $\Phi_i(x_0, t_0)$ stand for known quantities. We next show that the solution of equations (1.5) is unique. In point of fact, let us suppose these equations have two solutions P_1, Q_1, and P_2, Q_2. Then there will correspondingly be two solutions, u_1, v_1 and u_2, v_2, of the equations of motion for the half-plane. To the difference solution $u = u_1 - u_2$, $v = v_1 - v_2$, there will correspond the zero values of the initial data, of the body forces, of the stresses on the segment L', and also of the principal vector (X^0, Y^0) on L in the problem A, and of the vector $(X_k{}^0, Y_k{}^0)$ on L_k in the problem B. On L_k, this difference solution depends only on time:

$$u, \ v = \begin{cases} d_i(t) & \text{on } L_k \text{ in problem } A \\ d_{ik}(t) & \text{on } L_k \text{ in problem } B \end{cases}$$

and, in accordance with the difference stresses $P = P_1 - P_2$, $Q = Q_1 - Q_2$, this difference solution can be expressed in terms of the latter by the formulas [1]

$$2\pi\rho u(x_0, \ y_0, \ t_0) = \frac{\partial M}{\partial x_0} + \frac{\partial N}{\partial y_0}, \qquad 2\pi\rho v(x_0, \ y_0, \ t_0) = \frac{\partial M}{\partial y_0} - \frac{\partial N}{\partial x_0} \tag{1.6}$$

where

$$M = \iint_S [u_2^\circ Q(x, \ t) + v_1^\circ P(x, \ t)] \, dx dt, \qquad N = \iint_S [u_2^\circ Q(x, \ t) + v_2^\circ P(x, \ t)] \, dx dt \tag{1.7}$$

The fundamental solutions make it possible to enlarge the region of integration S somewhat and make it independent of x_0 and y_0. We can therefore easily justify the interchange of the order of differentiation and integration in (1.6), and obtain

$$\begin{aligned} 2\pi\rho u &= \iint_S \left[\left(\frac{\partial u_1^\circ}{\partial x_0} + \frac{\partial u_2^\circ}{\partial y_0}\right) Q(x, \ t) + \left(\frac{\partial v_1^\circ}{\partial x_0} + \frac{\partial v_2^\circ}{\partial y_0}\right) P(x, t)\right] dx \, dt \\ 2\pi\rho v &= \iint_S \left[\left(\frac{\partial u_1^\circ}{\partial y_0} - \frac{\partial u_2^\circ}{\partial x_0}\right) Q(x, \ t) + \left(\frac{\partial v_1^\circ}{\partial y_0} - \frac{\partial v_2^\circ}{\partial x_0}\right) P(x, \ t)\right] dx \, dt \end{aligned} \tag{1.8}$$

The region S of integration can be replaced by a set of rectangles of height $\theta_0 = t_0 - r_0/a$, where a is the propagation velocity of the longitudinal wave constructed on segment L_k. The equation (1.8) then takes the form

$$\begin{aligned} 2\pi\rho u &= \int_0^{\theta_0} \left\{ \int_L \left[\left(\frac{\partial u_1^\circ}{\partial x_0} + \frac{\partial u_2^\circ}{\partial y_0}\right) Q(x, \ t) + \left(\frac{\partial v_1^\circ}{\partial x_0} + \frac{\partial v_2^\circ}{\partial y_0}\right) P(x, \ t)\right] dx \right\} dt \\ 2\pi\rho v &= \int_0^{\theta_0} \left\{ \int_L \left[\left(\frac{\partial u_1^\circ}{\partial y_0} - \frac{\partial u_2^\circ}{\partial x_0}\right) Q(x, \ t) + \left(\frac{\partial v_2^\circ}{\partial y_0} - \frac{\partial v_2^\circ}{\partial x_0}\right) P(x, \ t)\right] dx \right\} dt \end{aligned} \tag{1.9}$$

We note that the difference solution u, v and its derivatives are zero at the moment t_0 at all points of the half-plane where $r_0 > at_0$.

Let B be a finite region bounded by the contour L in the x, y-plane, and let n be the exterior normal to L. Then

$$\frac{d}{dt}(T+V)=\iint\limits_B\left(X\frac{\partial u}{\partial t}+Y\frac{\partial v}{\partial t}\right)dxdy+\int\limits_L\left(X_n\frac{\partial u}{\partial t}+Y_n\frac{\partial v}{\partial t}\right)de \tag{1.10}$$

Here T is the kinetic and V the potential energy of the elastic medium:

$$T=\frac{1}{2}\iint\limits_B\rho\left[\left(\frac{\partial u}{\partial t}\right)^2+\left(\frac{\partial v}{\partial t}\right)^2\right]dx\,dy$$

$$V=\iint\limits_B\left[\frac{1}{2}\lambda(\varepsilon_x+\varepsilon_y)^2+\mu(\varepsilon_x{}^2+\varepsilon_y{}^2+2\gamma^2{}_{xy})\right]dxdy \tag{1.11}$$

Let us construct the region B_R in the half-plane $y \geqslant 0$ under consideration as follows. With the origin as center we describe a semicircle L_R with a radius R so large that all segments L_k will lie on its diameter inside the semicircle. Applying (1.10) to the region B_R and to the difference solution, we obtain

$$\frac{d}{dt}(T+V)=\int\limits_{L_R}\left(X_n\frac{\partial u}{\partial t}+Y_n\frac{\partial v}{\partial t}\right)dl \tag{1.12}$$

We will show that for all t:

$$\lim\int\limits_{L_k}\left(X_n\frac{\partial u}{\partial t}+Y_n\frac{\partial v}{\partial t}\right)dl=0 \qquad \text{as } R\to\infty \tag{1.13}$$

Indeed, the integrand in (1.13) is equal to zero at all points of the half-plane where $R > c + at_0$ (here c is the larger of the numbers $|a_1|$, $|b_n|$). Hence equation (1.13) is also valid at any moment at all points of the half-plane $T + V =$ const.

But since the initial data are equal to zero for the difference solution, it follows that $T + V = 0$. Thus,

$$\frac{\partial u}{\partial t}=\frac{\partial v}{\partial t}=\varepsilon_x=\varepsilon_y=\gamma_{xy}=0 \tag{1.14}$$

that is, the difference solution corresponds to the rigid displacement of the body. The stresses corresponding to this solution are zero; therefore, the difference stresses are zero at the points of the boundary of the half-plane:

$$P=P_1-P_2=0, \qquad Q=Q_1-Q_2=0 \tag{1.15}$$

The uniqueness theorem is thus proved.

2. Integral equation of the problem. Subsequently we will confine ourselves to considering the problem under the following boundary conditions: on the entire boundary $y = 0$ the tangential stresses τ_{xy} are given; on the part L of the boundary the displacements v are given, while on L' the stresses σ_y are given. We write the second of equations (1.4) in the form

$$2\pi\int_0^{t_0}(t_0-t)\left(\tau_{xy}-2\mu\frac{\partial v}{\partial x_0}\right)dt = \iint_S (u_2^\circ\tau_{xy}+v_2^\circ\sigma_y)\,dxdt + N_1(x_0,\ y_0,\ t_0) \tag{2.1}$$

where

$$N_1(x_0,\ y_0,\ t_0) = -\iiint_T (u_2^\circ X + v_2^\circ Y)\,d\tau +$$

$$+\rho\iint_{S_1}\left(u\frac{\partial u_2^\circ}{\partial t}+v\frac{\partial v_2^\circ}{\partial t}-u_2^\circ\frac{\partial u}{\partial t}-v_2^\circ\frac{\partial u}{\partial t}\right)dxdy$$

Letting the point M_0 approach a point P_0 within the region $\{y = 0,\ x\in L,\ 0\leqslant t<\infty\}$, where τ_{xy} and v are given, for σ_y we obtain the integral equation

$$\iint_{S_0}\sigma_y(x,\ t)\lim_{y_0\to 0} v_2^\circ\,dxdt = N_2(x_0,\ 0,\ t_0) \tag{2.2}$$

where N_2 includes known quantities which can be computed from the initial and boundary conditions, while S is the limiting value of the common part of region S and region D which is the set of all rectangles of height t_0 constructed on segment L_k. It is easy to evaluate the kernel of the equation (2.2):

$$\lim v_2^\circ = K_0(x-x_0,\ t_0-t) = \frac{4}{b^2}\operatorname{Re}\int_0^{\Theta}\frac{i\xi\sqrt{a^{-2}-\xi^2}}{F(\xi)}\,d\xi \qquad \left(\Theta=\frac{t_0-t}{x_0-x}\right) \tag{2.3}$$

where b is the velocity of propagation of the transverse wave and $F(\xi)$ is Rayleigh's function. We note certain properties of kernel (2.3). It is singular with a singularity of order $(x - x_0)^{-1}$ in the neighborhood of x_0; it has a logarithmic infinity for values of θ equal to the roots of Rayleigh's function, and is equal to zero on the boundary and outside the region S^0. This latter circumstance makes it possible to write equation (2.2) in the form

$$\int_0^{t_0}\int_L\frac{K_1(|x-x_0|,\ t_0-t)}{x-x_0}P(x,\ t)\,dxdt = N_2(x_0,\ 0\ \ t_0) \tag{2.4}$$

where the kernel

$$K_1(|x-x_0|,\ t_0-t) = \begin{cases}(x-x_0)K_0(x-x_0,\ t_0-t) & \text{в } S^\circ\\ 0 & \text{в } D-S^\circ\end{cases} \tag{2.5}$$

is bounded in the neighborhood of x_0.

The integral equation (2.4) of the Volterra type is of fundamental importance for the given problem.

3. Reduction to the Singular Equation. The validity of all following operations is assumed in order to obtain the solution function. This, of course, will necessitate an ultimate verification of the assumptions. We rewrite the equation (2.4) as

$$\int_L \frac{1}{x-x_0}\int_0^{t_0} K_1(|x-x_0|,\ t_0-t)\,P(x,\ t)\,dt\,dx = N_2 \tag{3.1}$$

Applying the Laplace transform and making use of the convolution theorem, we obtain an equivalent singular equation for the problem

$$\int_L \frac{K_1(|x-x_0|,\ s)}{x-x_0}\,P(x,\ s)\,dx = f(x_0,\ s) \quad (s=\sigma+i\tau,\ \sigma\geqslant\sigma_0>0) \tag{3.2}$$

Here $K_1(|x-x_0|,\ s)$, $P(x,\ s)$, $f(x_0,\ s)$ are the transforms of the functions $K_1(|x-x_0|,\ t)$, $P(x,\ t)$, $N_2(x_0,\ 0,\ t)$. It is easily verified that $K_1(0,\ s)=Ms^{-1}$, where M is a known constant whole value can easily be found.

We rewrite the equation (3.2) in the form

$$\frac{1}{\pi i}\int_L \frac{P(x,\ s)}{x-x_0}\,dx + \frac{1}{\pi i}\int_L K(x_0,\ x,\ s)\,P(x,\ s)\,dx = f(x_0,\ s) \tag{3.3}$$

Here

$$K(x_0,\ x,\ s) = \frac{K_1(|x-x_0|s) - K_1(0,\ s)}{K_1(0,\ s)\,(x-x_0)}$$

When $\lambda=\mu$ (Poisson's hypothesis), the following expression can be given for the kernel

$$K(x_0,\ x,\ s)\,K(0,\ s) = \operatorname{sign}(x-x_0)\Big\{\sum_{i=1}^{2} A_0{}^i\int_{\vartheta}^{\infty}\frac{u\chi(t)-t}{u}\,e^{-st}dt + \sum_{i,k=1}^{2} A_{ik}\int_{\vartheta_i}^{\infty}\operatorname{arc\,tg}\frac{\chi(t)}{g_{ik}}\,e^{-st}\,dt + \sum_{i=1}^{2} A_3{}^i\int_{\vartheta_i}^{\infty}\ln\left|\frac{\chi(t)-h}{\chi(t)+h}\right| e^{-st}dt - \sum_{i=1}^{2} A_0{}^i\int_0^{\vartheta} te^{-st}\,dt\Big\} \qquad \left(\theta_i=\frac{u}{a_i}\right) \tag{3.4}$$

$$\chi_i(t) = \sqrt{\frac{t^2}{u^2}-\frac{1}{a_i{}^2}}, \qquad h_i = \sqrt{\frac{1}{c^2}-\frac{1}{a_i{}^2}}, \qquad g_{ik} = \sqrt{\frac{1}{a_i{}^2}-\frac{1}{\alpha_k{}^2}}$$

where $u=|x-x_0|$.

Here $A_0{}^i$, A_{ik}, $A_3{}^i$ are known constants, s is the velocity of propagation of Rayleigh's wave, and $a_1 = a$, $a_2 = b$, $\alpha_1 = \alpha$, $\beta_1 = \beta$, α^{-2}, β^{-2} are real roots of the function

$$F_1(\xi) = 4\xi^2 \sqrt{a^{-2} - \xi^2}\sqrt{b^{-2} - \xi^2} - (b^{-2} - 2\xi^2)^2$$

It is not difficult to prove that

$$K(x_0, x_0, s) = \lim \frac{K_1(u, s) - K_1(0, s)}{(x - x_0) K(0, s)} = 0 \quad \text{as } x \to x_0 \tag{3.5}$$

and to show on the basis of (3.4) that the kernel $K(x_0, x, s)$ on L satisfies the Hoelder condition with $\nu < 1$.

Regularizing equation (3,3), we obtain [2]

$$P(x_0, s) + K^*KP(x, s) = K^*f + P_{n-1}(x_0, s) Z(x_0) \tag{3.6}$$

Here $Z(x_0)$ is a canonical solution of the given class and $P_{n-1}(x_0, x)$ is a polynomial of degree $n - 1$ whose coefficients depend on s_1:

$$K^*f = \frac{Z(x_0)}{\pi i} \int_L \frac{f\, dx}{Z^+(x)(x - x_0)}, \qquad K^*KP = \frac{1}{\pi i} \int_L N(x_0, x, s) P(x, s)\, dx$$

$$N(x_0, x, s) = \frac{Z(x_0)}{\pi i} \int_L \frac{K(x_1, x, s)\, dx_1}{Z^+(x_1)(x_1 - x_0)} \tag{3.7}$$

The homogeneous equation (3.5) has no roots different from zero (this is a consequence of the proved uniqueness theorem). Hence there exists one solution of equation (3.5), and one only. By means of the resolvent $R(x_0, x, s)$, this solution can be written in the form

$$P(x_0, s) = \Big[K^*f + \int_L R_1(x_0, x, s) K^*f dx + C_1(s)\Big(x_0{}^{n-1} +$$

$$+ \int_L R_1(x_0, x, s) x^{n-1} Z(x)\, dx\Big) + \ldots + C_n(s)\Big(1 + \int_L R_1(x_0, x, s) Z(x)\, dx\Big)\Big] Z(x_0)$$

$$R_1(x_0, x, s) = \frac{R(x_0, x, s)}{Z(x_0)} \tag{3.8}$$

The constants $C_k(s)$ are found from the auxiliary conditions of problem A and B. In problem B, the forces $P_k(t)$ are given on each segment L_k. In the image space we obtain

$$P_k(s) = \int_{L_k} P(x, s)\, dx \qquad (k = 1, \ldots, n) \tag{3.9}$$

In the problem A, all d_{2k} in formula (1.1), taken on the segments L_k, are equal,

$$d_{21}(s) = d_{22}(s) = \ldots = d_{2n}(s) \tag{3.10}$$

and we are given the entire pressure on all segments

$$P(s) = \int_L P(x,\ s)\,dx \tag{3.11}$$

Substituting (3.8) into (3.9), or making use (as in the static case) of conditions (3.10) and (3.11), we obtain (in either problem) a system of equations for the unknowns $C_k(s)$. This system has a unique solution which follows from the established uniqueness theorem.

The resolvent $R_1(x_0,\ x,\ s)$, considered as a function of s, has no singularities distinct from those of the kernel

$$N_1(x_0,\ x,\ s) = \frac{N(x_0,\ x,\ s)}{Z(x_0)}$$

This resolvent is, therefore, holomorphic in the half-plane Re $s = \sigma > \sigma_0 > 0$, and is bounded at infinity because the kernel N has this property. As is to be seen from the expression for N_1, the behavior of this function of s is determined by the behavior of the kernel

$$K(x_0,\ x,\ s) = \frac{s^2 M^{-1} K_1(u,\ s) - 1}{x - x_0}, \qquad u = |x - x_0|$$

where, in accordance with (3.3),

$$K_1(u,\ s) = -u \exp\frac{-su}{a} \int_0^\infty \int_{a^{-1}}^{\tau_1} \frac{\xi m(\xi,\ a)\, n(\xi,\ b)}{FF_1}\, d\xi e^{-s}\, dt -$$

$$- u \exp\frac{-su}{b} \int_0^\infty \int_{b^{-1}}^{\tau_2} \frac{4\xi^2 m^2(\xi,\ a)\, m(\xi,\ b)}{FF_1}\, d\xi e^{-st}\, dt \tag{3.12}$$

Here

$$m(\xi,\ r) = \sqrt{\xi^2 - \frac{1}{r^2}}, \quad \eta(\xi,\ r) = \frac{1}{r^2} - 2\xi^2$$

$$\tau_1 = \frac{1}{a} + \frac{t}{u}, \quad \tau_2 = \frac{1}{b} + \frac{t}{u}$$

If $u = 0$, namely, $x = x_0$, then $K_1 = Ms^2$, and $K(x_0,\ x_0,\ s) = 0$ for all s. Let $u \neq 0$. We will show that

$$\Pi_1(s,\ u) = s^2 u \int_0^\infty \int_{a^{-1}}^{t_1} \frac{\xi m(\xi,\ a)\, n^2(\xi,\ b)}{FF_1}\, d\xi e^{-st}\, dt \tag{3.13}$$

remaining uniformly bounded, and tends to zero as $|s|$ increases. Indeed, setting $t_1 = st$, we obtain

$$\Pi_1(s,\ u) = su\int_0^\infty \Big[\int_{a^{-1}}^{\vartheta_1} \frac{\xi m(\xi,\ a)\, n^2(\xi,\ b)}{FF_1}\, d\xi \Big] e^{-t}\, dt \qquad \left(\vartheta_1 = \frac{1}{a} + \frac{t}{su}\right) \tag{3.14}$$

Here the integration has to be carried out along the ray $\arg t_1 = \arg s = \phi = \text{const} \neq 0$. However, it is easy to show that this path of integration can be replaced by one which passes along the real axis, if the function F_1 has no imaginary zeros. Assuming $\zeta = s$, let us estimate $|\pi_1(\zeta^{-1},\ u)|$ for small values of ζ. For the path of integration selecting a segment of a ray starting at the origin,

$$\frac{t}{u}|\zeta| = \frac{t}{u}\rho, \qquad \rho = |\zeta|, \qquad \zeta_1 = \zeta - \frac{1}{a}$$

we have

$$\Big| \int_0^{tu^{-1}\zeta} \frac{(\zeta_1 + a^{-1})\,[b^{-2} - 2(\zeta_1 + a^{-1})^2]^2 \sqrt{\zeta_1^2 - 2a^{-1}\xi_1}}{F(a^{-1} + \xi_1)\, F_1(a^{-1} + \xi_1)}\, d\xi_1 \Big| \leqslant \max|(\ldots)| \left|\frac{t}{u}\right| |\zeta| \tag{3.15}$$

The parentheses $(\ldots)$ indicate the integrand, and we take the maximum of its absolute value on the segment indicated. Since

$$\max|(\ldots)| \leqslant \max\Big| \frac{(tu^{-1}\xi_1 + a^{-1})\,[b^{-2} - 2(tu^{-1}\xi_1 + a^{-1})^2]^2}{F(tu^{-1}\xi_1 + a^{-1})\, F_1(tu^{-1}\xi_1 + a^{-1})} \sqrt{\frac{t}{u}} \sqrt{\frac{t}{u}\xi_1 + \frac{1}{a}} \Big| \sqrt{\rho} \tag{3.16}$$

and since the expression standing between the absolute value signs is bounded by, say, L, we have

$$\left|\Pi_1\left(\frac{1}{\zeta},\ u\right)\right| \leqslant L\sqrt{\rho}\int_0^\infty te^{-t}\, dt = L\sqrt{\rho} \tag{3.17}$$

Thus the term $\pi_1(\zeta^{-1},\ u)$ approaches zero as $\sqrt{\rho}$ goes to zero, and remains uniformly bounded. The same properties are possessed by the coefficient of $\exp(-sb^{-1}u)$ in the second term in expression (3.12), and hence by function $K_1(u,\ s)$. Consequently $s \to \infty$ may be taken as the limit under the integral sign of the integral determining N_1. Since the limiting value is finite, the boundedness of the kernel $N_1(x_0,\ x,\ s)$ with $s = \infty$ has thus been proved. In consequence of the boundedness of the resolvent R_1, the first two terms in formula (3.6) and the function K^*F belong to the class of functions that can be represented by means of a Laplace integral. It follows from this that all the $C_k(s)$ belong to this class. Thus the function $P(x_0,\ s)$ given by formula (3.8) can be represented by means of a Laplace integral and by using a Mellin transform we can find the original

$$P(x_0, t_0) = \frac{1}{2\pi i} \int_{\sigma_0 - i\infty}^{\sigma_0 + i\infty} P(x_0, s) e^{st_0} ds \qquad (3.18)$$

Let us verify that function $P(x_0, t_0)$ is a solution of (2.4). In point of fact function $P(x_0, s)$, being a solution of (3.5), is a solution of the equivalent equation (3.2).

Let $c_j (j = q + 1, \ldots, m)$ be the values which $Z(x)$ takes on at infinity. Let us set

$$Z(x) = Z_1(x) Z_0(x), \quad (Z_1(x) = \sum_{j=q+1}^{m} (x - c_j)^{\gamma_j} \qquad (-1 < \operatorname{Re} \gamma_j < 0))$$

where $Z_0(x)$ is a function bounded in the neighborhoods of the c_j. Then $P(x, s) = Z_1(x) P_1(x, s)$, where $P_1(x, s)$ is a function bounded near the c_j and belongs to the class H_ϵ. The equation (3.2) can be written in the form

$$\int_L \frac{Z_1(x) K_1(|x - x_0|, s)}{x - x_0} P_1(x, s) dx = f_1(x, s) \qquad (3.19)$$

We will establish the uniform convergence of the integral

$$\frac{1}{2\pi i} \int_{\sigma_0 - i\infty}^{\sigma_0 + i\infty} K_1(|x - x_0|, s') P_1(x_1 s) e^{st} ds \, (s = \sigma_0 + i\tau, \sigma_0 > 0) \qquad 3.20)$$

After division by the factor $e^{\sigma_s t}$, we have

$$\left| \frac{1}{2\pi} \int_{\sigma_0 - i\infty}^{\sigma_0 + i\infty} K_1(|x - x_0|, s) P_1(x, s) e^{i\tau t} d\tau \right| \leqslant$$

$$\leqslant \int_{-\infty}^{\infty} \frac{|s^2 K_1(|x - x_0|, s)|| P_1(x_1 s)}{|s^2|} d\tau \leqslant \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{A(s) B(s)}{|s|^3} d\tau \qquad (3.21)$$

because $|s^2 k_1(|x - x_0|, s)| \leqslant A(s)$, $|P_1(x, s)| < B(s)$, where $A(s)$ and $B(s)$ are bounded nonnegative functions. Hence, taking the Mellin transform of both parts of (3.20) and changing the order of integration, we obtain

$$\int_L \frac{Z_1(x)}{x - x_0} \int_0^{t_0} K_1(|x - x_0|, t_0 - t) P_1(x, t) dx dt = N_2(x_0, 0, t_0) \qquad (3.22)$$

or

$$\int_0^{t_0} \int_L \frac{K_1(|x - x_0|, t_0 - t) P_1(x, t)}{x - x_0} dx dt = N_2(x_0, 0, t_0) \qquad (3.23)$$

Function (3.18) is thus seen to be in fact a solution of our problem. The existence theorem has thus been proved.

In conclusion we note that if $t \to \infty$, and if during this process the functions tend to definite limits, then, owing to (3.8), on the boundary we obtain the familiar solution of the static problem on the pressure of stamps on an elastic half-plane.

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Translated by H.P.T.